Statistical inference

lecture1

Definitions & Order Statistics

Lecturer

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Chapter one

Def (1) let r.v's $X_1, X_2, \dots, X_n$ be indep. and each r.v has the common p.d.f $f(x)$ a random simple of size n from a distribution has p.d.f

Def (2) Any function of observations $X_1, \dots, X_n$ which contain unknown parameter θ is called statistic i.e. No μ, σ^2

Def (3) Any statistic $T(x)$ that is used to estimate some function of the unknown parameter θ, $g(\theta)$ is called an estimtor and The value of the statistic is called an estimate

order statistic

let $X_1, \dots, X_n$ be a r.v's from $f(x, \theta)$, $\theta \in \Omega$ continuous r.v define

Y_1 smallest of $(X_1, \dots, X_n)$

Y_2 second smallest of $(X_1, \dots, X_n)$

$\vdots$

Y_n Largest of $(X_1, \dots, X_n)$

The order value $Y_1 \leq Y_2 \leq - \leq Y_n$

The joint p.d.f of $Y_1, -, Y_n$ is

$a < y_1 < y_2 < - < b$

$$g(y_1, -, y_n) = \begin{cases} n!\, f(y_1), -, f(y_n) & \\ 0 & o/w \end{cases}$$

The distribution function of the J the order statistic when $J = 1, 2, -, n$

$$g(y_j) = \frac{n!}{(j-1)!\,(n-j)!} \left[F(y_j)\right]^{j-1} \left[1 - F(y_j)\right]^{n-j} \cdot f(y_j)$$

where

$$f(y_j) = f(x) \quad , \quad F(y_j) = \int_{-\infty}^{y_j} f(x)\, dx$$

The joint p.d.f of y_i, y_j is

$$g(y_i, y_j) = \begin{cases} \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} \left[F(y_i)\right]^{i-1} \left[F(y_j) - F(y_i)\right]^{j-i-1} \\ \left[1 - F(y_j)\right]^{n-j} \cdot f(y_i)\, f(y_j) \end{cases}$$

where $a < y_i < y_j < b$

Example Let $Y_1 < Y_2 < Y_3$ the order statistic of r.v's of size 3 from the Uniform dist.

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} & 0 < x < \theta \\ 0 & o/w \end{cases}$$

Find ① $g(y_1)$ ② $g(y_2)$ ③ $E(y_2)$

Sol

① $$g(y_j) = \frac{n!}{(j-1)!\,(n-j)!} \left[F(y_j)\right]^{j-1} \left[1-F(y_j)\right]^{n-j} f(y_j)$$

$$f(y_1) = f(x) = \frac{1}{\theta}$$

$$F(y_1) = \int_0^{y_1} \frac{1}{\theta}\,dx = \frac{x}{\theta}\Big|_0^{y_1} = \frac{y_1}{\theta}$$

$$g(y_1) = \frac{3!}{(1-1)!\,(3-1)!} \left[1-\frac{y_1}{\theta}\right]^{3-1} \left[\frac{y_1}{\theta}\right]^{1-1} \cdot \frac{1}{\theta}$$

$$= \frac{3}{\theta^3}(\theta - y_1)^2 \qquad 0 < y_1 < \theta$$

② $g(y_2) = f(x) = \frac{1}{\theta}$

$$F(y_2) = \int_0^{y_2} \frac{1}{\theta}\,dx = \frac{y_2}{\theta}$$

$$g(y_2) = \frac{3!}{(2-1)!\,(3-2)!}\left[\frac{y_2}{\theta}\right]^{2-1}\left[1-\frac{y_2}{\theta}\right]^{3-2}\cdot\frac{1}{\theta}$$

$$= \frac{6}{\theta^3}\, y_2(\theta - y_2) \qquad 0 < y_2 < \theta$$

(3) $E(y_2) = \int_0^\theta y_2\, g(y_2)\, dy_2$

$$= \frac{6}{\theta^3}\int_0^\theta y_2^2(\theta - y_2)\, dy_2$$

$$= \frac{6}{\theta^3}\int_0^\theta (\theta y_2^2 - y_2^3)\, dy_2$$

$$= \frac{6}{\theta^3}\left[\theta\frac{y_2^3}{3} - \frac{y_2^4}{4}\Big|_0^\theta\right]$$

$$= \frac{6}{\theta^3}\left[\frac{\theta^4}{3} - \frac{\theta^4}{4}\right)$$

$$= 2\theta - \frac{3}{2}\theta$$

$\therefore E(y_2) = \frac{\theta}{2}$

Example :- let $Y_1 < Y_2 < Y_3$ be order stat. of a r.v of size 3 from

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & 0 < x < \infty \\ 0 & o/w \end{cases}$$

find ① $g(y_1)$ ② $g(y_2)$ ③ $g(y_3)$

④ $E(2y_2)$ ⑤ $var(y_2)$

Sol

① H.w

② $g(y_j) = \frac{n!}{(j-1)!\,(n-j)!} \left[F(y_j)\right]^{j-1} \left[1 - F(y_j)\right]^{n-j} \cdot f(y_j)$

$f(y_2) = f(x) = \frac{1}{\theta} e^{-\frac{y_2}{\theta}}$

$F(y_2) = \int_0^{y_2} f(x)\,dx = \int_0^{y_2} \frac{1}{\theta} e^{-\frac{x}{\theta}}\,dx$

$= \left[-e^{-\frac{x}{\theta}}\right]_0^{y_2}$

$= -\left(e^{-\frac{y_2}{\theta}} - e^{-\frac{0}{\theta}}\right)$

$= 1 - e^{-\frac{y_2}{\theta}}$

$$g(y_2) = \frac{3!}{(2-1)!\,(3-2)!}\left[1-e^{-\frac{y_2}{\theta}}\right]^{2-1}\left[1-(1-e^{-\frac{y_2}{\theta}})\right]^{3-2}\cdot\frac{1}{\theta}e^{-\frac{y_2}{\theta}}$$

$$= \frac{6}{\theta}\left[1-e^{-\frac{y_2}{\theta}}\right]e^{-\frac{y_2}{\theta}}\cdot e^{-\frac{y_2}{\theta}}$$

$$= \frac{6}{\theta}\left[1-e^{-\frac{y_2}{\theta}}\right]e^{-\frac{2y_2}{\theta}}$$

$$\therefore g(y_2) = \frac{6}{\theta}\left(e^{-\frac{2y_2}{\theta}} - e^{-\frac{3y_2}{\theta}}\right)$$

(3) H.w

(4) $E(y_2) = \int_0^\infty y_2\, g(y_2)\, dy_2$

$$= \int_0^\infty y_2 \cdot \frac{6}{\theta}\left(e^{-\frac{2y_2}{\theta}} - e^{-\frac{3y_2}{\theta}}\right) dy_2$$

$$= \frac{6}{\theta}\left[\int_0^\infty y_2\, e^{-\frac{2y_2}{\theta}} dy_2 - \int_0^\infty y_2\, e^{-\frac{3y_2}{\theta}} dy_2\right]$$

$$\therefore E(y_2) = \frac{6}{\theta}\left[\frac{\theta^2}{4}\underbrace{\int_0^\infty \frac{4}{\theta^2}e^{-\frac{2y_2}{\theta}}y_2\,dy_2}_{=1} - \frac{\theta^2}{9}\underbrace{\int_0^\infty \frac{9}{\theta^2}e^{-\frac{3y_2}{\theta}}y_2\,dy_2}_{=1}\right]$$

$$= \frac{6}{\theta}\left[\frac{\theta^2}{4} - \frac{\theta^2}{9}\right] = \frac{3}{2}\theta - \frac{2}{3}\theta$$

$$\therefore E(y_2) = \frac{5}{6}\theta$$

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Note from Gamma Dist.

$$\therefore f(x,\lambda) = \frac{\lambda^r}{\Gamma r} x^{r-1} e^{-\lambda x}$$

where

$$y_2' = x^{r-1} \Rightarrow \therefore 1 = r-1 \Rightarrow r = 2$$

$$e^{-\lambda x} = e^{-\frac{2y_2}{\theta}} \Rightarrow \lambda = \frac{2}{\theta}$$

$$\because r = 2 \therefore \lambda^r = \left(\frac{2}{\theta}\right)^r$$

$$\therefore \lambda^2 = \left(\frac{2}{\theta}\right)^2 = \frac{4}{\theta^2}$$

$$e^{-\lambda x} = e^{-\frac{3y_2}{\theta}} \Rightarrow \lambda = \frac{3}{\theta}$$

$$\therefore \lambda^2 = \left(\frac{3}{\theta}\right)^2 = \frac{9}{\theta^2}$$

$$\because X \sim \text{Gamma}(r,\lambda)$$

$$\therefore \int_0^\infty f(x,\lambda) = 1$$

هنا تم مشابهة هذه الصيغة لسهولة الحل وتحويلها إلى دالة احتمالية

(5) $var(y_2)$ H.w