Statistical inference

lecture2

Chapter Two: Estimation

Properties of good estimator

Lecturer

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Chapter Two

1-1 Estimation

1-2 Properties of good Estimator

1-2-1 unbiasness

Def: Let $x_1, x_2 \dots x_n$ be a r.v from a population having $f(x, \theta)$ with unknown θ

let $\hat{\theta}$ be an estimator of θ, therefor $\hat{\theta}$ is to be unbised estimator of θ iff $E(\hat{\theta}) = \theta$.

otherwise, the bise B of $\hat{\theta}$ can be defined as:

$$B = E(\hat{\theta}) - \theta$$

1. $\hat{\theta}$ is an unbised estimator for θ iff

$$E(\hat{\theta}) = \theta$$

2. $T(x)$ is an unbised estimator for $g(\theta)$ iff

$$E(T(x)) = g(\theta)$$

Example 1 : $X \sim N(\mu, \sigma^2)$, is $\bar{X}$ an unbised estimator for θ?

Sol (1) : $E(\bar{X}) = \mu$?

$$E(\bar{x}) = E\left(\frac{\sum x_i}{\cancel{n}}\right) = \frac{E(x_1 + \cdots + x_n)}{n}$$

$$E(\bar{x}) = \frac{E x_1 + \cdots E x_n}{n} = \frac{\mu + \mu + \cdots + \mu}{n} = \frac{\cancel{n}\mu}{\cancel{n}}$$

$\mu = \theta$ $\quad\therefore E(\bar{x}) = \mu$

2- Is $s^2 = \frac{\sum (x_i - \bar{x})^2}{n}$ an unbised estimator for σ^2?

$E(S^2) = \sigma^2$?

$$E(S^2) = E\left[\frac{\sum (x_i - \bar{x})^2}{n}\right] = E\left[\frac{\sum x_i^2}{n} - \frac{2\bar{x}\sum x_i}{n} + \frac{\cancel{n}\bar{x}^2}{\cancel{n}}\right]$$

$$= E\left[\frac{\sum x_i^2}{n} - \bar{x}^2\right] = \frac{\cancel{\sum} E x_i^2}{\cancel{n}} - E(\bar{x})^2$$

$$= E x^2 - E(\bar{x})^2$$

$$\boxed{\begin{array}{l} \text{note: } \sigma^2 = Ex^2 - (E(x))^2 \Rightarrow Ex^2 = \sigma^2 + \mu^2 \\ V(\bar{x}) = E\bar{x}^2 - (E\bar{x})^2 \Rightarrow \frac{\sigma^2}{n} = E\bar{x}^2 + \mu^2 \end{array}}$$

$$\therefore E S^2 = (\sigma^2 + \mu^2) - \left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= \sigma^2 - \frac{\sigma^2}{n} \Rightarrow \sigma^2\left(1 - \frac{1}{n}\right)$$

$$\therefore ES^2 = \sigma^2\left(\frac{n-1}{n}\right)$$

$\therefore S^2$ is abise estimator for σ^2. $E\left(\frac{n}{n-1} S^2\right) = \sigma^2$

$$E\left(\frac{n}{n-1}\cdot\frac{\Sigma(X_i-\bar{X})^2}{n}\right)=\sigma^2$$

$$E\left(\frac{\Sigma(X_i-\bar{X})^2}{n-1}\right)=\sigma^2$$

$\therefore \dfrac{\Sigma(x_i-\bar{x})^2}{n-1}$ is unbised estimator for σ^2

3- Is $\bar{X}^2$ an unbised estimator for M^2?

$$V(\bar{x})=E(\bar{x})^2-(E(\bar{x}))^2$$

$$E(\bar{x})^2=V(\bar{x})+(E(\bar{x}))^2$$

$$=\frac{\sigma^2}{n}+M^2$$

$\therefore \bar{X}^2$ is abised for M^2

note: $T(x)$ اي دالة بدلالة x $g(\theta)$ اي دالة بدلالة θ

Example(2): Let $X_1,\dots,X_n$ be a r.v from $U(0,\theta)$

Let Y_n the largest of n observations

is Y_n an unbised estimator for θ?

Solu.1: $$g(y_j)=\frac{n!}{(j-1)!\,(n-j)!}\left[F(y_j)\right]^{j-1}\left[1-F(y_j)\right]^{n-j}\cdot f(y_j)$$

$$f(y_n)=f(x)=\frac{1}{\theta}$$

$$F(y_n)=\int_0^{y_n}\frac{1}{\theta}\cdot dx=\frac{Y_n}{\theta}$$

$$\therefore g(y_n)=\frac{n!}{(n-1)!\,(n-n)!}\left[\frac{y_n}{\theta}\right]^{n-1}\left[1-\frac{y_n}{\theta}\right]^{n-n}\cdot\frac{1}{\theta}$$

$\therefore g(y_n) = \frac{n}{\theta^n} y_n^{n-1}$

$E(y_n) = \int y_n g(y_n) \cdot dy_n$

$= \frac{n}{\theta^n} \int_0^\theta y_n^{n-1} \cdot y_n \cdot dy_n$

$= \frac{n}{\theta^n} \int_0^\theta y_n^n \, dy_n = \frac{n}{\theta^n} \frac{y_n^{n+1}}{n+1} \Big|_0^\theta$

$= \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1}$

$\therefore E(y_n) = \frac{n}{n+1}\theta$ $\therefore y_n$ is abise estimator for θ

2. Determine a multiple of Y_n that is estimator for θ :

we have $E(y_n) = \frac{n}{n+1}\theta$

$a\,E(y_n) = \theta$

$a\left(\frac{n}{n+1}\theta\right) = \theta \Longrightarrow a = \frac{n+1}{n}$

Example 3 :- Let $X_1, \dots X_n$ be a r.v's from a Population

having $f(x,\theta) = \frac{1}{\theta} e^{-x/\theta}$ $\theta > 0$

show that the sample mean $\bar{x}$ is an unbised est, for θ.

Sol :- $E(\bar{x}) = E\left(\frac{\sum x_i}{n}\right) = \frac{1}{n} \sum E\, x_i$

we know that $E(X) = \theta$

$\therefore E(\bar{x}) = \frac{1}{n} \sum \theta = \frac{1}{n} n\theta = \theta$ $\therefore \bar{x}$ is unbised

Example 4:: $X_1, \dots, X_n$ is a r.s from a Point Binomial dist is the sample mean $\bar{X}$ an unbised est. for the Parameter p?

Sol:: $E(\bar{X}) = E\left(\frac{\sum X_i}{n}\right)$

$$= \frac{1}{n} \sum (E X_i) = \frac{1}{n} \sum p$$

$$= \frac{1}{\cancel{n}} \cdot \cancel{n} p = p$$

H.W:: Let $X_1 \dots X_n$ be r.v with $f(x) = \frac{e^{-x} \lambda^x}{x_i}$

$x = 0, 1, \dots$

show that $\bar{X}$ is unbised est. for λ?

1-2-2 Mean square error::

Def:: A useful measure of Performance of an estimator $\hat{\theta}$ of θ is the average squared error $MSE(\theta)$ wich can be defined as::

$$MSE(\theta) = E(\hat{\theta} - \theta)^2 = Var(\hat{\theta}) + B^2$$

Example:. let $x_1, \dots x_n$ be a r.v from $B(1, \theta)$

use (MSE) to compare $\bar{x}$ and x_1 for θ

Sol:. 1. $E(\bar{x}) = E\left(\frac{1}{n} \sum x_i\right) = \frac{1}{n} \sum E x_i = M = \theta$

$E(x_1) = M = \theta$

$\therefore$ both $\bar{x}$ & x_1 are unbised est. for θ

2. $Var(\bar{x}) = Var\left(\frac{x_1 + x_2 \dots x_n}{n}\right)$

$= \frac{1}{n^2}\left[V(x_1) + \dots + V(x_n)\right] + 0$

$= \frac{n \sigma^2}{n^2} = \frac{\theta(1-\theta)}{n}$

$var(x_1) = \sigma^2 = \theta(1-\theta)$

$MSE(\bar{x}) = Var(\bar{x}) + B^2$

$= \frac{\theta(1-\theta)}{n} + 0$ $\Leftarrow$

note:. هنا B = صفر لأن
هنا أن $\bar{x}$ غير متحيز.

$= \frac{\theta(1-\theta)}{n}$

$MSE(x_1) = Var(x_1) + B^2$

$= \theta(1-\theta) + 0 = \theta(1-\theta)$

$\therefore MSE(\bar{x}) < MSE(x_1)$

$\therefore \bar{x}$ is be better than x_1