Statistical inference

lecture4

Chapter Two: Estimation

Examples

Lecturer

Omar Adil

Example(3) Let $X_1, \ldots, X_n$ be a r.v from a popul.

with p.d.f

$$f(x,\theta) = \frac{x}{\theta^2} e^{-\frac{x}{\theta}} , \quad \begin{matrix} 0 < \theta \\ 0 < x \end{matrix}$$

i) Is $\frac{\bar{X}}{2}$ an unbised est. for θ?

ii) Find $MSE(\bar{X})$?

iii) Is $\frac{\bar{X}}{3}$ a consistent est. for θ?

Sol

from p.d.f $X \sim Gamma(r, \lambda)$

$\therefore r = 2 \quad , \lambda = \frac{1}{\theta}$

$\therefore E(x) = \frac{r}{\lambda} = \frac{2}{\frac{1}{\theta}} = 2\theta$

$Var(X) = \frac{r}{\lambda^2}$

$Var(x) = \frac{2}{\frac{1}{\theta^2}}$

$Var(x) = 2\theta^2$

i) $\therefore E(\frac{\bar{X}}{2}) = \frac{1}{2} E(\bar{x})$

$= \frac{1}{2} E(\frac{X_1 + \cdots + X_n}{n})$

$= \frac{1}{2n} \cdot (EX_1 + \cdots + EX_n)$

$= \frac{1}{2\cancel{n}} \cdot \cancel{n} E(X)$

$= \frac{1}{\cancel{2}} \cdot \cancel{2}\theta \Rightarrow E(X) = \theta$

$\therefore \frac{\bar{X}}{2}$ is an unbised est. for θ

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ii) $MSE(\bar{x}) = var(\bar{x}) + B^2(\bar{x})$

$\because var(\bar{x}) = \frac{\sigma^2}{n}$

$= \frac{2\theta^2}{n}$

$E(\bar{x}) = 2\theta$

$\therefore B^2 = [E(\bar{x}) - \theta]^2$

$= [2\theta - \theta]^2$

$\therefore B^2 = \theta^2$

$\therefore MSE(\bar{x}) = var(\bar{x}) + B^2(\bar{x})$

$= \frac{2\theta^2}{n} + \theta^2$

$= \frac{2+n}{n}\theta^2$

iii)

$\lim_{n\to\infty} E(\frac{\bar{x}}{3}) = \lim_{n\to\infty} \frac{1}{3} \cdot 2\theta$

$= \frac{2}{3}\theta$

$\lim_{n\to\infty} var(\frac{\bar{x}}{3}) = \frac{1}{9} \lim_{n\to\infty} \frac{\sigma^2}{n}$

$= 0$

$\therefore \frac{\bar{x}}{3}$ is not consistent est. for θ

<u>H.W</u> find the consistent for $\frac{\bar{x}}{3}$ by $\lim_{n\to\infty} MSE(\frac{\bar{x}}{3})$

Example (4) Let $X_1, -, X_n$ be a r.v from $U(0, \theta)$

and let Y_n be the largest of (n) observation

i) Is Y_n an unbised est. for θ?

ii) Compute $MSE(Y_n)$

iii) Is Y_n consistent est. for θ?

Sol

i) $$g(y_j) = \frac{n!}{(j-1)!\,(n-j)!}\left[F(y_j)\right]^{j-1}\left[1-F(y_j)\right]^{n-j} f(y_j)$$

$$f(y_n) = f(x) = \frac{1}{\theta} \qquad 0 < y_n < \theta$$

$$F(y_n) = \int_0^{y_n} \frac{1}{\theta}\, dy_n$$

$$= \frac{y_n}{\theta}$$

$$\therefore g(y_n) = \frac{n!}{(n-1)!\,(n-n)!}\left[\frac{y_n}{\theta}\right]^{n-1}\left[1-\frac{y_n}{\theta}\right]^{n-n} \cdot \frac{1}{\theta}$$

$$= \frac{n}{\theta^n}\left[y_n\right]^{n-1}$$

$$\therefore E(g(y_n)) = \int_0^{\theta} y_n \frac{n}{\theta^n} y_n^{n-1}\, dy_n$$

$$= \frac{n}{\theta^n}\int_0^{\theta} y_n^n\, dy_n$$

$$= \frac{n}{\cancel{\theta^n}} \cdot \frac{\cancel{\theta^{n+1}}}{n+1} = \frac{n\theta}{n+1}$$ $\therefore \bar{y}_n$ is bise est. for θ

ii) $MSE(Y_n) = var(y_n) + B^2(y_n)$

$var(y_n) = E(y_n^2) - (E(y_n))^2$

$\Rightarrow E(y_n^2) = \int_0^{\theta} y_n^2 \frac{n}{\theta^n} y_n^{n-1} \, dy_n$

$= \frac{n}{\theta^n} \int_0^{\theta} y_n^{n+1} \, dy_n$

$= \frac{n}{\cancel{\theta^n}} \cdot \frac{\cancel{\theta^{n+2}}}{n+2}$

$E(y_n^2) = \frac{n\theta^2}{n+2}$

$\therefore var(y_n) = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2$

$\therefore MSE(y_n) = \left[\frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2\right] + \left[\frac{n\theta}{n+1} - \theta\right]^2$

$= \frac{n\theta^2}{n+2} - \cancel{\left(\frac{n\theta}{n+1}\right)^2} + \cancel{\left(\frac{n\theta}{n+1}\right)^2} - \frac{2n\theta^2}{n+1} + \theta^2$

$= \frac{n\theta^2}{n+2} - \frac{2n\theta^2}{n+1} + \theta^2$

iii) $\lim_{n\to\infty} MSE(Y_n) = \lim_{n\to\infty}\left[\frac{n\theta^2}{n+2} - \frac{2n\theta^2}{n+1} + \theta^2\right]$

$= \lim_{n\to\infty}\left[\frac{\theta^2}{1+\frac{2}{n}} - \frac{2\theta^2}{1+\frac{1}{n}} + \theta^2\right]$

$= \theta^2 - 2\theta^2 + \theta^2 = 0$

$\therefore y_n$ is consis. est. for θ

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