Statistical inference

lecture5

Chapter Two

Sufficiency

Lecturer

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sufficiency

① **Def** Let $x_1, x_2, \ldots, x_n$ be a r.v from p.d.f $f(x, \theta)$, $\theta \in \Omega$, A statistic $T(x)$ is defined be sufficient statistic for θ iff the conditional dist. of $x_1 \to x_n$ given the value of T doesnot invalue for all others value t of T

$$P(X_1 = x_1, X_2 = x_2 \to X_n = x_n \mid T = t) = \frac{P(X_1 = x_1, \to X_n = x_n, T = t)}{P(T = t)}$$

doesnot invalue θ

Example Let $x_1 \to x_n$ be a r.v from $B(1, \theta)$ show That $T = \Sigma x_i$ is suff. stat. for θ

Sol
$$P(x_1 \to x_n \mid T = t) = \frac{P(x_1 \to x_n, t)}{P(t)}$$

① $P(x_1 \to x_n, t) = \prod_{i=1}^{n} P(x_i, \theta)$

$$= \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}$$

$$= \theta^{\Sigma x_i} (1-\theta)^{n - \Sigma x_i}$$

$$= \theta^{t} (1-\theta)^{n-t}$$

(2 $p(t) = C_t^n \, \theta^t (1-\theta)^{n-t}$, $t = 0, 1, -$

$\therefore p(x_1, x_2, -, x_n \mid T = t) = \dfrac{\cancel{\theta^t (1-\theta)^{n-t}}}{C_t^n \cancel{\theta^t (1-\theta)^{n-t}}}$

$= \dfrac{1}{C_t^n}$

$\therefore \Sigma x_i$ is a suff. stat. for θ.

② Factorization theorem

Let $x_1, -, x_n$ be a r.v's of size n from p.d.f $f(x, \theta)$. A statistic $T(x)$ is suff. stat. for θ iff:

$f(x_1, -, x_n, \theta) = g(T(\underline{x}), \theta) \cdot h(\underline{x})$

where g is non-negative and Depend on x therogh $T(x)$ and h is non-negative and independ of θ.

Example let $x_1, \dots, x_n$ be a r.v from $B(1, \theta)$ show that $T = \sum x_i$ is suff. stat. for θ

Sol $L = P(x_1, \dots, x_n, \theta) = \prod_{i=1}^{n} P(x_i, \theta)$

$$= \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}$$

$$= \theta^{\sum x_i} (1-\theta)^{n - \sum x_i} \prod_{i=1}^{n} I_{(0,1)}^{(x)}$$

$$= \theta^{\sum x_i} \frac{(1-\theta)^n}{(1-\theta)^{\sum x_i}} \prod_{i=1}^{n} I_{(0,1)}^{(x)}$$

$$\therefore L = \underbrace{\left(\frac{\theta}{1-\theta}\right)^{\sum x_i} (1-\theta)^n}_{g(T(x), \theta)} \cdot \prod_{i=1}^{n} I_{(0,1)}^{(x)}$$

$\therefore T = \sum x_i$ is suff. stat. for θ

Example Let $x_1, \dots, x_n$ a r.v with $N(\theta, \sigma^2)$ where σ^2 is known, is $\bar{X}$ suff. stat. for θ

Sol $L = \prod_{i=1}^{n} f(x_i, \theta)$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\theta)^2}{2\sigma^2}}$$

$$\therefore L = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum(x_i-\theta)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum(x_i-\bar{x}+\bar{x}-\theta)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum(x_i-\bar{x})^2}{2\sigma^2} - \frac{n(\bar{x}-\theta)^2}{2\sigma^2}}$$

$$= \underbrace{e^{-\frac{\cancel{n}(\bar{x}-\theta)^2}{2\sigma^2}}}_{g(T(\underline{x}),\theta)} \underbrace{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum(x_i-\bar{x})^2}{2\sigma^2}}}_{h(\underline{x})}$$

$\therefore T(x) = \bar{x}$ is suff. stat. for θ

③ Exponential Family:-

Let x has p.d.f, $f(x,\theta)$, $\theta \in \Omega$, θ is a single parameter. A family of density $f(x,\theta)$ is called member of exponential family if can be wirtten in following form

$$f(x,\theta) = Exp\left\{ T(x)\, g(\theta) + D(\theta) + \delta(x) \right\} \cdot I_{(-\infty,\infty)}^{(x)}$$

where $T(x)$, $g(\theta)$, $D(\theta)$ and $\delta(x)$ are suitable functions.

<u>Example:-</u> Let $x \sim B(n, \theta)$, $0 < \theta < 1$

is X a suff. stat. for θ

<u>Sol</u> $\Rightarrow$

$$f(x,\theta) = C_x^n \, \theta^x (1-\theta)^{n-x}$$

$$= Exp\left\{ Ln\, C_x^n + x\, Ln\, \theta + (n-x)\, Ln(1-\theta) \right\}$$

$$= Exp\left\{ Ln\, C_x^n + x\, Ln\, \theta + n\, Ln(1-\theta) - x\, Ln(1-\theta) \right\}$$

$$= Exp\left\{ \underbrace{Ln\, C_x^n}_{\delta(x)} + \underbrace{x\, Ln \frac{\theta}{1-\theta}}_{T(x)\cdot g(\theta)} + \underbrace{n\, Ln\, \theta}_{D(\theta)} \right\}$$

$\therefore$ Binomal is amember of EXP. family

$\therefore$ X is suff. stat. for θ

Remark

under the r.v's $x_1, -, x_n$ if $f(x, \theta)$ is member of Exp. family Then $\sum T(x)$ is a suff. stat. for θ

Example $\because X \sim B(n, \theta)$

$$T(x) = X \Rightarrow \sum T(x) = \sum x_i$$

$\therefore \sum x_i$ is suff. stat. for θ

Example if $X \sim U(0, \theta)$

is uniform Dist. a member of Exp. family

①

$$f(x, \theta) = \frac{1}{\theta} I_{(0, \theta)}^{(x)}$$

$$= \operatorname{Exp}\{-\operatorname{Ln}\theta\} I_{(0, \theta)}^{(x)}$$

$\therefore$ uniform is not a member of Exp family

② by using def.

$$P(x_1, \to x_n | T = t) = \frac{L}{g}$$

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$\therefore L = \left(\frac{1}{\theta}\right)^n \prod_{i=1}^{n} I_{(0,\theta)}^{(x)}$

$g(t) = 0 < y_1 < \cdots < y_n < \theta$

$\therefore t = \hat{\theta} = y_n$ is a m.l.e of θ

$\therefore g(y_n)$ found by using order statistic

(3) Factorization therom

$$L = \underbrace{\left(\frac{1}{\theta}\right)^n \prod_{i=1}^{n} I_{(0,\theta)}^{(x)}}_{g(T(x),\theta)} \cdot \frac{1}{h(x)}$$

$0 < y_1 < y_2 < \cdots < y_n < \theta$

Example let $x_1, \ldots, x_n$ be a r.v's from a p.d.f $f(x,\theta) = \theta e^{-\theta x}$ $x, \theta > 0$

Show that $\sum x_i$ is a suff. stat. for θ

sol (1) by using def.

$f(x_1, \ldots, x_n \mid T = t) = \frac{L}{g}$

$L = \prod_{i=1}^{n} f(x_i, \theta)$

$= \prod_{i=1}^{n} \theta e^{-\theta x}$

$$\therefore L = \theta^n e^{-\theta \Sigma x_i}$$

$$\because M_x^{(t)} = E e^{tx}$$

$$= \frac{\theta}{\theta - t}$$

$$\therefore M_{\Sigma x_i}^{(t)} = E e^{t \Sigma x_i}$$

$$= E e^{t(x_1 + x_2 + \cdots + x_n)}$$

$$= E e^{t x_1} \cdot E e^{t x_2} \cdots E e^{t x_n}$$

$$= \frac{\theta}{\theta - t} \cdot \frac{\theta}{\theta - t} \cdots \frac{\theta}{\theta - t}$$

$$= \left(\frac{\theta}{\theta - t}\right)^n$$

$$\therefore \Sigma x_i \sim \text{gamma}(n, \theta)$$

$$g(t) = \frac{\theta^n}{\Gamma n} \Sigma x^{n-1} e^{-\theta \Sigma x_i}$$

$$\therefore f(\underline{x} \mid T = t) = \frac{\cancel{\theta^n} \cancel{e^{-\theta \Sigma x_i}}}{\frac{\cancel{\theta^n}}{\Gamma n} \Sigma x_i^{n-1} \cancel{e^{-\theta \Sigma x_i}}}$$

$$= \frac{\Gamma n}{\Sigma x_i^{n-1}}$$

$\therefore \Sigma x_i$ is suff. stat. for θ 8

H.W Suppose that $X_1, \ldots, X_n$ is a r.v from the dist. with

$$f(x,\theta) = \theta^2 x e^{-\theta x}$$

find the suff. stat. for the parameter θ