Statistical inference

lecture6

Chapter Two

Efficiency

And

Cramer-Rao variance

Lecturer

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5- Efficiency

① when $\hat{\theta}_1, \hat{\theta}_2$ is unbised, Then

$MSE(\hat{\theta}_1, \hat{\theta}_2) = Var(\hat{\theta}_1, \hat{\theta}_2)$, i.e.

$$e(\hat{\theta}_1, \hat{\theta}_2) = \frac{Var(\hat{\theta}_1)}{Var(\hat{\theta}_2)}$$

IF $e(\hat{\theta}_1, \hat{\theta}_2) = 1$, That Means $\hat{\theta}_1$ like $\hat{\theta}_2$

IF $e(\hat{\theta}_1, \hat{\theta}_2) > 1$, That Means $\hat{\theta}_2$ better than $\hat{\theta}_1$

IF $e(\hat{\theta}_1, \hat{\theta}_2) < 1$, That Means $\hat{\theta}_1$ better than $\hat{\theta}_2$

② when one of $\hat{\theta}_1, \hat{\theta}_2$ is unbiase and other biase then

$$e(\hat{\theta}_1, \hat{\theta}_2) = \frac{MSE(\hat{\theta}_1)}{MSE(\hat{\theta}_2)}$$

6- Minimum Variance bound Estimator

Let $X_1, X_2, \dots, X_n$ be a random sample from a dist. with p.d.f $f(x,\theta)$, where $\theta \in \Omega$.

If it is possible to express

$\frac{\partial}{\partial \theta} \ln f(x,\theta)$ in the following form

$$\frac{\partial}{\partial \theta} \ln f(x,\theta) = A(\theta)\{T(x) - g(\theta)\}$$

or

$$\frac{\partial}{\partial \theta} \ln L(\theta, x) = A(\theta)\{T(x) - g(\theta)\}$$

Then $T(x)$ is a M.V.B.E for $g(\theta)$ with variance $\frac{g'(\theta)}{A(\theta)}$

Example :-

Let $X_1, \dots, X_n$ be a r.v from dist. with p.m.f $f(x,\theta) = \theta^x (1-\theta)^{1-x}$, $x = 0, 1$

Find the M.V.B.E for θ

Sol

$$L(\theta, x_1 \ldots, x_n) = \prod_{i=1}^{n} P(x, \theta)$$

$$= \prod_{i=1}^{n} \theta^{x} (1-\theta)^{1-x}$$

$$= \theta^{\Sigma x_i} (1-\theta)^{n-\Sigma x_i}$$

$$\therefore \ln L(\theta) = \Sigma x \ln(\theta) + (n - \Sigma x_i) \ln(1-\theta)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{\Sigma x_i}{\theta} - \frac{n - \Sigma x_i}{1-\theta}$$

$$= \underbrace{\frac{n}{\theta(1-\theta)}}_{A(\theta)} \left\{ \underbrace{\frac{\Sigma x_i}{n}}_{T(x)} - \underbrace{\theta}_{g(\theta)} \right\}$$

$\therefore T(x) = \bar{x}$ is M.V.B.E for θ.

To find $var(\bar{x}) = \dfrac{g'(\theta)}{A(\theta)}$

$$= \frac{1}{\dfrac{n}{\theta(1-\theta)}}$$

$$\therefore var(\bar{x}) = \frac{\theta(1-\theta)}{n}$$

Example

In sampling from $N(0, \sigma^2)$, Show That $\frac{1}{n}\sum x_i^2$ is a M.V.B.E of σ^2

Sol

$\because X \sim N(0, \sigma^2)$

$$\therefore f(x, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_i^2}{2\sigma^2}}$$

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_i^2}{2\sigma^2}}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{\sum x_i^2}{2\sigma^2}}$$

$$\ln L = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{\sum x_i^2}{2\sigma^2}$$

$$\frac{\partial \ln L}{\partial \sigma^2} = \frac{-\frac{n}{2}\cancel{2\pi}}{\cancel{2\pi}\sigma^2} + \frac{\sum x_i^2}{2\sigma^4}$$

$$= \frac{-n\sigma^2 + \sum x_i^2}{2\sigma^4}$$

$$= \underbrace{\frac{n}{2\sigma^4}}_{A(\theta)} \left\{ \underbrace{\frac{\sum x_i^2}{n}}_{T(x)} - \underbrace{\sigma^2}_{g(\theta)} \right\}$$

$\therefore T(x) = \frac{\sum x_i^2}{n}$ is a M.V.B.E for σ^2

$$\therefore \operatorname{var}\left(\frac{\sum x_i^2}{n}\right) = \frac{g'(\theta)}{A(\theta)}$$

$$= \frac{1}{\frac{n}{2\sigma^4}}$$

$$= \frac{2\sigma^4}{n}.$$

Now IS $\operatorname{var}\left(\frac{\sum x_i^2}{n}\right) = \frac{2\sigma^4}{n}$??

$\because X \sim N(0, \sigma^2)$, $Z = \frac{X-0}{\sigma} \sim N(0,1)$

$\therefore Z^2 = \frac{X^2}{\sigma^2} \sim \chi^2_{(1)}$

$\therefore \sum Z^2 = \frac{\sum x_i^2}{\sigma^2} \sim \chi^2_{(n)}$

So $\frac{\sum x_i^2}{n\sigma^2} \sim \chi^2_{(n)}/n$

$\therefore \frac{\sum x_i^2}{n} \sim \frac{\sigma^2}{n} \chi^2_{(n)}$

$$\therefore \operatorname{var}\left(\frac{\sum x_i^2}{n}\right) = \operatorname{var}\left(\frac{\sigma^2}{n} \chi^2_{(n)}\right)$$

$$= \frac{\sigma^4}{n\cancel{{}^2}} \cdot 2\cancel{n}$$

$$= \frac{2\sigma^4}{n}$$

7- Cramer - Rao inequality

Let $x_1, --, x_n$ be a r.v from population with p.d.f $f(x, \theta)$, where $\theta \in \Omega$.
Let $T = t(x_1, x_2 \rightarrow x_n)$ be an unbiased estimator of $g(\theta)$. Then subject To Certain regularity condition:

$$Var(T) \geq \frac{[g'(\theta)]^2}{-nE\left[\frac{\partial^2}{\partial \theta^2} \ln f(X, \theta)\right]}$$

or

$$Var(T) = \frac{[g'(\theta)]^2}{-E\left[\frac{\partial^2}{\partial \theta^2} \ln L(\theta, x)\right]}$$

Example

Let $x \sim P(\theta)$, is $\bar{X}$ an effici. est. for θ? show that $\bar{X}$ is the M.V.B.E of θ with variance $\frac{\theta}{n}$

sol

① $eff = \frac{var(C.R)}{var(\bar{x})}$

$\because var(\bar{x}) = \frac{\sigma^2}{n} = \frac{\theta}{n}$

$var(T) \geq \frac{[g'(\theta)]^2}{-E\left[\frac{\partial^2}{\partial \theta^2} LnL\right]}$

$\because x \sim P(\theta)$

$\therefore P(x, \theta) = \frac{e^{-\theta} \theta^{x}}{x_i!}$

$L(\theta, x) = \prod_{i=1}^{n} \frac{e^{-\theta} \theta^{x}}{x_i!}$

$= \frac{e^{-n\theta} \theta^{\Sigma x_i}}{\Pi x_i!}$

$LnL = -n\theta + \Sigma x_i \ln \theta - \Sigma \ln x_i!$

$\frac{\partial LnL}{\partial \theta} = -n + \frac{\Sigma x_i}{\theta}$

$\frac{\partial^2 LnL}{\partial \theta^2} = \frac{-\Sigma x_i}{\theta^2}$

$$\therefore E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = E\left[\frac{-\Sigma x_i}{\theta^2}\right]$$

$$= \frac{-\Sigma E(x_i)}{\theta^2}$$

$$= \frac{-n\cancel{\theta}}{\theta\cancel{^2}}$$

$$= \frac{-n}{\theta}$$

$$\because g(\theta) = \theta \Rightarrow g'(\theta) = 1$$

$$\therefore Var(T) \geq \frac{[1]^2}{-\left[\frac{-n}{\theta}\right]} = \frac{\theta}{n}$$

$$\therefore Var(C.R) = \frac{\theta}{n}$$

$$\therefore eff = \frac{Var(C.R)}{Var(\bar{x})}$$

$$= \frac{\theta/n}{\theta/n} = 1$$

$\therefore$ $\bar{x}$ is an eff. est. for θ

(2) To find M.V.B.E for θ

$$\because \frac{\partial \ln L}{\partial \theta} = -n + \frac{\sum x_i}{\theta}$$

$$= \frac{-n\theta + \sum x_i}{\theta}$$

$$= \underset{A(\theta)}{\frac{n}{\theta}} \left\{ \underset{T(x)}{\frac{\sum x_i}{n}} - \underset{g(\theta)}{\theta} \right\}$$

$T(x) = \bar{x}$ is a M.V.B.E

$$\therefore var(\bar{x}) = \frac{g'(\theta)}{A(\theta)}$$

$$Var(\bar{x}) = \frac{1}{n/\theta} = \frac{\theta}{n}$$

Example

Let $x_1, -, x_n$ be a r.v's with p.d.f

$$f(x,\theta) = \frac{2}{\theta} e^{-\frac{2}{\theta} x}$$

(1) what is samllest possible variance for an unbiased estimator of Mean of this dist.

(2) can you find an unbiased est. has this variance?

Sol ① To find smallest variance we must find $Var(T) \geq \dfrac{[g'(\theta)]^2}{-nE\left[\dfrac{\partial^2 \ln f}{\partial \theta^2}\right]}$

$\because X \sim Exp\left(\frac{2}{\theta}\right)$

$\therefore Mean = E(X) = \dfrac{1}{\frac{2}{\theta}}$

$= \dfrac{\theta}{2}$

$\therefore g(\theta) = \dfrac{\theta}{2} \implies g'(\theta) = \dfrac{1}{2}$

$\ln f(x, \theta) = \ln 2 - \ln \theta - \dfrac{2x}{\theta}$

$\dfrac{\partial \ln f(x, \theta)}{\partial \theta} = \dfrac{-1}{\theta} + \dfrac{2x}{\theta^2}$

$\dfrac{\partial^2 \ln f(x, \theta)}{\partial \theta^2} = \dfrac{1}{\theta^2} - \dfrac{2x(2\cancel{\theta})}{\theta^{\cancel{4}}}$

$= \dfrac{1}{\theta^2} - \dfrac{4x}{\theta^3}$

$\therefore E\left[\dfrac{\partial^2 \ln f(x, \theta)}{\partial \theta^2}\right] = \dfrac{1}{\theta^2} - \dfrac{4Ex}{\theta^3}$

$= \dfrac{1}{\theta^2} - \dfrac{4\left(\frac{\theta}{2}\right)}{\theta^3}$

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$$\therefore E\left[\frac{\partial^2 \ln f(x,\theta)}{\partial \theta^2}\right] = \frac{1}{\theta^2} - \frac{2}{\theta^2}$$

$$= \frac{-1}{\theta^2}$$

$$\therefore Var(T) \geq \frac{[1/2]^2}{-n[-1/\theta^2]}$$

$$\geq \frac{1/4}{n/\theta^2} = \frac{\theta^2}{4n}$$

② $E(\bar{x}) = \mu = \frac{\theta}{2}$

$$\therefore Var(\bar{x}) = \frac{\sigma^2}{n} = \frac{(\theta/2)^2}{n}$$

$$= \frac{\theta^2}{4n}$$

$\therefore \bar{x}$ has that samllest variance.

<u>H.w</u> find the M.V.B.E for $\frac{\theta}{2}$

<u>H.w</u> A sample of size (n) is drawn from $B(n, \theta)$, find ① M.V.B.E for θ

② Is it efficient ?

H.W

Let $X_1, \ldots, X_n$ be a r.v with p.m.f

$$P(x, \theta) = \frac{e^{-\frac{1}{\theta}}}{\theta^x \, x!} \quad , \; x = 0, 1, \ldots$$

① Find C.R.L.B for $\frac{1}{\theta}$

② Can you find an unbiased esti. that has this variance

③ find M.V.B.E for $\frac{1}{\theta}$